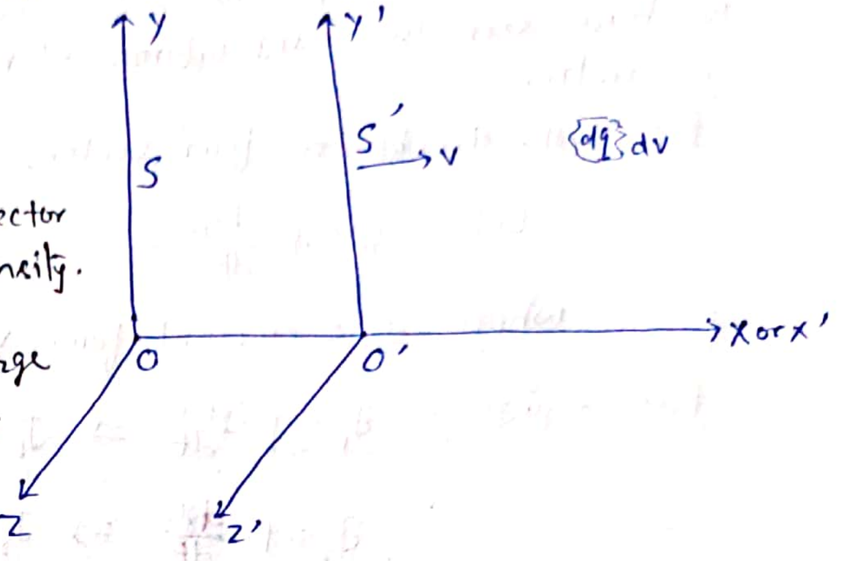
Current Four vector : Lorentz Transformation Equations for charge density (ρ) and current density ($\vec{J}$)



Equation of continuity is

$$\operatorname{div} \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

where $\vec{J}$ = current density vector
ρ = volume charge density.

current density $\vec{J}$ and charge density ρ are not separate entities in relativistic case. ρ and $\vec{J}$ are grouped to-gether as

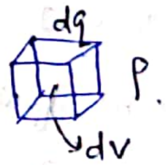
$$J_\mu = (\vec{J}, ic\rho) \quad \text{——— (A)} \quad \mu = 1, 2, 3, 4$$

where J_μ is known as current four vector

$\vec{J} = J_1 \hat{i} + J_2 \hat{j} + J_3 \hat{k}$ and $J_4 = ic\rho$ = 4rth component of J_μ.

Here result (A) is directly written. Now we have to justify the result (A).

Justification :- Consider a small volume element of volume dV containing charge dq.



For elementary volume dV, volume charge density ρ is supposed to be constant.

$$\text{so} \quad dq = \rho . dV \quad \text{——— (1)}$$

Multiplying both sides by dx_μ where x_μ = position four vector

$$dq . dx_\mu = \rho . dx_\mu . dV$$

$$\Rightarrow dq . dx_\mu = \rho . \frac{dx_\mu}{dt} . dV . dt \quad \text{——— (2)}$$

Here dq is scalar and dx_μ is four vector so the product $dq . dx_\mu$ will be four vector. Since L.H.S of eqn (2) is four vector so R.H.S of eqn (2) must be four vector.

Now $dV \cdot dt = dx \cdot dy \cdot dz \cdot dt$ $\quad \because dV = dx \cdot dy \cdot dz$

$dV \cdot dt = dx_1 \cdot dx_2 \cdot dx_3 \cdot dt$ = 4D volume element.

We have seen that 4D volume element is Lorentz invariant or scalar.

For R.H.S to be a four vector, $\rho \cdot \frac{dx_\mu}{dt}$ must be four vector.

Let $J_\mu = \rho \cdot \frac{dx_\mu}{dt}$ ——— (3)

where J_μ = current four vector, $\mu = 1, 2, 3, 4$.

From eqn (3), $J_1 = \rho \cdot \frac{dx_1}{dt} \Rightarrow J_1 = \rho \cdot V_1$ ——— 4(a)

$J_2 = \rho \cdot \frac{dx_2}{dt} \Rightarrow J_2 = \rho \cdot V_2$ ——— 4(b)

$J_3 = \rho \cdot \frac{dx_3}{dt} \Rightarrow J_3 = \rho \cdot V_3$ ——— 4(c)

and $J_4 = \rho \cdot \frac{dx_4}{dt} = \rho \frac{d(ict)}{dt}$ $\quad \because x_4 = ict$

$\Rightarrow J_4 = \rho ic \cdot 1 \Rightarrow J_4 = ic\rho$ ——— 4(d)

Thus components of current four vector are

$J_1 = \rho V_1$, $J_2 = \rho V_2$, $J_3 = \rho V_3$ & $J_4 = ic\rho$ ——— (4)

So current four vector J_μ is

$$J_\mu = (J_1, J_2, J_3, J_4)$$

or $J_\mu = (\vec{J}, ic\rho)$

where $\vec{J} = J_1\hat{i} + J_2\hat{j} + J_3 \cdot \hat{k}$ and $J_4 = ic\rho$

<u>Lorentz Transformation of current four vector or Lorentz transformation equations for charge density (ρ) and current density ($\vec{J}$)</u>

Since all four vectors follow Lorentz transformation so current four vector will also follow Lorentz transformation.

Lorentz transformation of current four vector is given by

$$J'_\mu = a_{\mu\nu} \cdot J_\nu \quad \text{——— (1)}$$

where J'_μ = components of current four vector in frame S'

J_ν = components of current four vector in frame s.

$\alpha_{\mu\nu}$ = Transformation matrix (4×4 square matrix)

$$\alpha_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad \begin{matrix} \mu = 1, 2, 3, 4. \\ \nu = 1, 2, 3, 4. \end{matrix}$$

Using matrix form of equⁿ ①, we get

$$\begin{pmatrix} J_1' \\ J_2' \\ J_3' \\ J_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \times \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix} \quad \text{—— (2)}$$

On solving equⁿ ②, we get

$$J_1' = \gamma \cdot J_1 + 0 \cdot J_2 + 0 \cdot J_3 + i\beta\gamma \cdot J_4 \Rightarrow J_1' = \gamma(J_1 + i\beta J_4)$$

$$\Rightarrow J_1' = \gamma(J_1 + i \cdot \tfrac{v}{c} \cdot ic\rho) \qquad \because \beta = \tfrac{v}{c}, \; J_4 = ic\rho$$

$$\Rightarrow J_1' = \gamma(J_1 - v\rho) \text{ —— } 3(a) \qquad \because i^2 = -1$$

Again $J_2' = 0 \cdot J_1 + 1 \cdot J_2 + 0 \cdot J_3 + 0 \cdot J_4$

$$\Rightarrow J_2' = J_2 \text{ —— } 3(b)$$

Again $J_3' = 0 \cdot J_1 + 0 \cdot J_2 + 1 \cdot J_3 + 0 \cdot J_4$

$$\Rightarrow J_3' = J_3 \text{ —— } 3(c)$$

Again $J_4' = -i\beta\gamma \cdot J_1 + 0 \cdot J_2 + 0 \cdot J_3 + \gamma \cdot J_4$

$$\Rightarrow J_4' = \gamma(J_4 - i\beta J_1) \text{ —— } 3(d_1)$$

or $ic\rho' = \gamma(ic\rho - i\frac{v}{c} \cdot J_1) \qquad \because J_4' = ic\rho', \; J_4 = ic\rho, \; \beta = \frac{v}{c}$

$$\Rightarrow \rho' = \gamma(\rho - \tfrac{v}{c^2} J_1) \text{ —— } 3(d_2)$$

Thus Lorentz transformation of current four vector (charge density ρ and current density $\vec{J}$) will be

$$\boxed{J_1' = \gamma(J_1 - \rho \cdot v), \; J_2' = J_2, \; J_3' = J_3 \;\&\; J_4' = \gamma(J_4 - i\beta J_1) \text{ or } \rho' = \gamma(\rho - \tfrac{v}{c^2} J_1)} \quad \text{—— (4)}$$

* **Inverse Lorentz Transformation of current four vector:**

For obtaining Inverse Lorentz transformation of current four vector, we interchange primed and unprimed quantities and put $-v$ in place of v in eqn (4),

Inverse Lorentz transformation of current four vector will be

$$\boxed{J_1 = \gamma(J_1' + \rho' v), \quad J_2 = J_2', \quad J_3 = J_3' \quad \text{and} \quad J_4 = \gamma(J_4' + i\beta J_1') \quad \text{or} \quad \rho = \gamma\left(\rho' + \frac{v}{c^2} J_1'\right)} \quad \text{—— (5)}$$

* **Equation of continuity in terms of current four vector**

As we know that equation of continuity is

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{—— (6)}$$

Now $\vec{\nabla}\cdot\vec{J} = \left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)\cdot(J_1\hat{i} + J_2\hat{j} + J_3\hat{k})$

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} \quad \text{—— (7)}$$

Again $\dfrac{\partial \rho}{\partial t} = \dfrac{\partial (ic\rho)}{\partial (ict)} = \dfrac{\partial J_4}{\partial x_4}$ —— (8) $\quad \because J_4 = ic\rho,\ x_4 = ict$

Using eqns (7) and (8) in eqn (6), we get

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0$$

$$\Rightarrow \boxed{\frac{\partial J_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \Box\cdot J_\mu = 0} \quad \text{—— (9)}$$

It is equation of continuity in terms of current four vector.

$\Box$ is a box operator

$\Box \equiv \dfrac{\partial}{\partial x_\mu}$ = 4D divergence operator.

$\Box = \dfrac{\partial}{\partial x_1}\hat{i} + \dfrac{\partial}{\partial x_2}\hat{j} + \dfrac{\partial}{\partial x_3}\hat{k} + \dfrac{\partial}{\partial x_4}\cdot\hat{p} \rightarrow$ 4D box operator.

* If charge distribution be at rest in the frame S then
$\vec{J} = 0 \Rightarrow J_1 = J_2 = J_3 = 0$ & $J_4 = ic\rho$.
Then components of current four vector in the frame S′ will be

$$J_1' = \gamma(0 - \rho.v) \Rightarrow J_1' = -\gamma\rho v$$
$$J_2' = 0$$
$$J_3' = 0$$

and $J_4' = \gamma(J_4 - i\beta.0) = -\gamma J_4 = +\gamma.ic\rho \Rightarrow J_4' = i\gamma c\rho$
or $ic\rho' = i\gamma c\rho \Rightarrow \rho' = \gamma\rho$

Thus $\boxed{J_1' = -\gamma\rho v = -\rho' v,\ J_2' = 0,\ J_3' = 0 \ \&\ J_4' = \gamma ic\rho \text{ or } \rho' = \gamma\rho}$

Therefore, if charge be at rest in the frame S then current or current density in the frame S will be zero but it produces a convection current in the frame S′.

i.e, $dq' = dq$

* Electric charge is relativistic invariant but electric charge density (ρ) is not relativistic invariant because $\rho' = \gamma\rho$ i.e, $\rho' \neq \rho$.

$$dq' = \rho' dv' = \gamma\rho.\frac{dv}{\gamma} \qquad \because \rho' = \gamma\rho \text{ and } dv' = \frac{dv}{\gamma}.$$

$$\Rightarrow dq' = \rho.dv$$

$$\Rightarrow \boxed{dq' = dq}$$

Thus electric charge is relativistic invariant